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THE CUMULATIVE EFFECT OF RANDOM LOSSES IN A TRANSMISSION LINE

by

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0. Summary

We will find the probability distribution of losses from reflections at discontinuities in a transmission line, on the assumption that the reflection coefficients at the discontinuities have random phase, and, in addition, random magnitudes associated with manufacturing uncertainties (tolerances).

1. Introduction

Suppose a transmission line has n discontinuities with associated voltage standing wave ratios (VSWR's) $r_1 \leq r_2 \leq r_3 \leq \ldots \leq r_n$. A common procedure for describing the cumulative effect of reflections at the discontinuities is to say that the resulting VSWR will be $r_1 r_2 r_3 \ldots r_n$ when the mismatches add in the worst phase, or $r_n / r_1 r_2 \ldots r_{n-1}$ when they combine in the best phase. (See, for example, [4], p.35, 611). This gives a rather large interval which is of limited use to designers. Can a more precise description of the overall VSWR be given in probability terms? In [3], Mullen and Pritchard give such a description, based on the approximation that reflection coefficients are additive. We will develop a procedure based on combining the reflection coefficients in a more precise way.

2. Combining Reflection Coefficients

Let $\gamma_1, \gamma_2, \ldots, \gamma_n$ be the reflection coefficients associated with n discontinuities in a transmission line. We are interested in the overall reflection coefficient γ. We assume that $\gamma_1, \ldots, \gamma_n$ can not be given exactly, and hence that each is a random complex number with a distribution that will be specified. We want the probability distribution of γ, and in particular, that of $|\gamma|$, which will give an indication of the total reflection losses in the system.

Now a rough approximation to γ is $\gamma \cong \gamma_1 + \ldots + \gamma_n$, as was considered in [3]. We will use a more precise relation between γ and the γ_k's and find the probability distribution of γ from that of the γ_k's. The relation we want is

$$\frac{\gamma}{1-\gamma} = \frac{\gamma_1}{1-\gamma_1} + \frac{\gamma_2}{1-\gamma_2} + \ldots + \frac{\gamma_n}{1-\gamma_n} .$$

This is equivalent to adding the shunt admittances of the discontinuities. See, for example, Montgomery, [2], p.71.

If we let these generalized admittances be

$$\eta = \gamma/(1-\gamma), \eta_1 = \gamma_1/(1-\gamma_1), \ldots, \eta_n = \gamma_n/(1-\gamma_n),$$

then

$$\eta = \eta_1 + \eta_2 + \ldots + \eta_n$$

and $\gamma = \eta/(1+\eta)$. We will assume that $\eta_1, \ldots, \eta_n$ are independent random complex variables, and hence, by the central limit theorem, we can expect that the real and complex parts of η will be jointly normally distributed. We then must find the distribution of $|\gamma|$, when $\gamma = \eta/(1+\eta)$ and the distribution of η is the bivariate normal.

3. Distribution of the Admittances

Our procedure for finding the distribution of the overall reflection coefficient is this: we are given n reflection coefficients $\gamma_1, \gamma_2, \ldots, \gamma_n$, assumed to be independent random complex numbers. We first convert reflection coefficients, γ_k, to admittances, η_k, according to the relation $\eta_k = \gamma_k/(1 - \gamma_k)$. We then add the admittances, getting $\eta = \eta_1 + \eta_2 + \ldots + \eta_n$. By the central limit theorem, η will be approximately normally distributed in the complex plane. Then we convert η back to an overall reflection coefficient, γ, according to the relation $\gamma = \eta/(1 + \eta)$.

We assume that each reflection coefficient, γ_k, has the form

$$\gamma_k = c_k e^{iuk},$$

where c_k and u_k are independent random variables, c_k with some distribution on the interval $(0,1)$ and u_k with some symmetric distribution on the interval $(0,2\pi)$. From these assumptions, it follows (see Theorem 1, appendix) that the real and imaginary parts of η_k are uncorrelated. Thus the real and imaginary parts of $\eta = \eta_1 + \eta_2 + \ldots + \eta_n$ are uncorrelated, and since they may be taken as jointly normal, they will be independent, for independence and zero correlation are equivalent for normal variates.

Thus if we write

$$\eta = x + iy = \eta_1 + \eta_2 + \ldots + \eta_n$$

we may take x and y to be independent normal random variables with means and variances calculated from expression (2) in the appendix. In particular, when we assume that each u_k is uniformly distributed over $(0,2\pi)$, then we have

$$E(x) = E(y) = 0$$

$$\sigma_x^2 = \sigma_y^2 = \frac{1}{2} \Sigma E[c_k^2/(1 - c_k^2)].$$

4. <u>The Distribution of</u> $|\eta/(1 + \eta)|$ <u>from that of</u> η.

We have $\eta = x + iy$, where x and y are independent normal random variables, and want the distribution of $|\gamma|$, where $\gamma = \eta/(1 + \eta)$. Presumably, $|\gamma|$ takes values between 0 and 1, and for $0 < t < 1$, we note that $|\gamma| < t$ if and only if, in the complex plane, the distance from η to the origin is less than t times the distance from η to $-1 + 0i$. In terms of x and y, this condition may be put in the form

$$P[|\gamma| < t] = P[(x - t^2/(1 - t^2)) + y^2 < (t/(1 - t^2))^2].$$

Thus the probability that $|\gamma|$ will be less than t is the bivariate normal probability measure of an offset circle. The standard bivariate normal probability measure, $(\sigma_x = \sigma_y = 1)$, of a circle with center distance d from the origin and radius r, designated $C(d,r)$,

$$C(d,r) = P[(\frac{x}{\sigma} + d)^2 + (\frac{y}{\sigma})^2 \leq r^2]$$

is a well known function. It may be computed by the formula

$$C(d,r) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} F_k(r^2)$$

where

$$\lambda = d^2/2, \quad \text{and, if} \quad \mu = r^2/2,$$

$$F_k(r^2) = 1 - e^{-\mu} - \mu e^{-\mu} - \frac{\mu^2}{2!} e^{-\mu} - \dots - \frac{\mu^k}{k!} e^{-\mu}$$

Tables of $C(d,r)$ are available from the author, [1], or from Rand, [5].

Our results may be summarized as follows:

If a transmission line has discontinuities with reflection coefficients $\gamma_1 = c_1 e^{iu_1}, \gamma_2 = c_2 e^{iu_2}, \dots, \gamma_n = c_n e^{iu_n}$, where the c's and u's are independent, each c_k with some distribution on the interval $(0,1)$, and each u_k uniformly distributed on the interval $(0,2\pi)$, then the resulting reflection coefficient, γ, related to $\gamma_1,\dots,\gamma_n$ by

$$\frac{\gamma}{1-\gamma} = \frac{\gamma_1}{1-\gamma_1} + \frac{\gamma_2}{1-\gamma_2} + \dots + \frac{\gamma_n}{1-\gamma_n},$$

has magnitude with probability distribution

$$\text{(1)} \qquad P[|\gamma| < t] = C\left[\frac{t^2}{\sigma(1-t^2)}, \frac{t}{\sigma(1-t^2)}\right].$$

where

$$\sigma^2 = \frac{1}{2}\sum_1^n E[c_k^2/(1-c_k^2)]$$

and $C(d,r)$ is the standard normal probability measure of a circle with radius r, center distance d from the origin.

Graphs of the distribution function and density function of $|\gamma|$, for $\sigma = .02, .04, \dots, .26$, are drawn in figure 1.

5. Determining $E[c^2/(1 - c^2)]$.

We have seen that the probability distribution of the magnitude of the resultant reflection coefficient, on the assumption that the phases of the component reflection coefficients are uniformly distributed, is given by (1). It depends on a single parameter, σ, related to the individual reflection coefficients $\gamma_k = c_k e^{iu_k}$ by

$$\sigma^2 = \frac{1}{2}\sum_1^n E[c_k^2/(1 - c_k^2)].$$

It is therefore important that we determine the expected value of $c_k^2/(1 - c_k^2)$ for each k. In most practical problems, c will be given in terms of a tolerance, e.g., $c = .10 \pm .04$. How can we determine $E(c^2/(1 - c^2)$ when all we know is that c is given in a form such as $c = .10 \pm .04$? Fortunately, we can get good upper and lower bounds on this expectation for two reasons: The function $x^2/(1 - x^2)$ is convex, and the range of the typical c is small. The bounds we want are as follows: If c is given by $c = c_0 \pm \delta$, where c_0 is the expected value of c, then no matter how c is distributed on the interval $c_0 - \delta < c < c_0 + \delta$, we have

$$\frac{c_0^2}{1 - c_0^2} \leq E(\frac{c^2}{1 - c^2}) \leq \frac{1}{2}[\frac{L^2}{1 - L^2} + \frac{U^2}{1 - U^2}]$$

where $L = c_0 - \delta$ is the lower, and $U = c_0 + \delta$ the upper, tolerance value on c. In the example above, $c = .10 \pm .04$, we have

$$\frac{.10^2}{1 - .10^2} \leq E(\frac{c^2}{1 - c^2}) \leq \frac{1}{2}[\frac{.06^2}{1 - .06^2} + \frac{.14^2}{1 - .14^2}]$$

or

$$.0101 \leq E(\frac{c^2}{1 - c^2}) \leq .0118.$$

Thus we have contained our unknown expected value in a rather small interval.

6. Examples

Example 1. Suppose we have 8 discontinuities, each with VSWR rated $1.1 \pm .04$. Conventional theory states that the worst mismatch will result in a VSWR of about $(1.1)^8 = 2.14$, or a reflection coefficient with magnitude $(2.14 - 1)/(2.14 + 1) = .36$. Figure 2 gives the actual distribution of the resultant $|\gamma|$, it is clear that only very rarely will $|\gamma|$ be anywhere near $.36$. To describe the possible variation of $|\gamma|$ in probability terms, we first convert tolerances on each VSWR to tolerances on the magnitude of the corresponding reflection coefficient. If r is in the range $1.06 < r < 1.14$ then $c = (1 - r)/(1 + r)$ is in the range $.029 < c < .065$, which we write as $c = .048 \pm .018$, taking $.048$ as the expected value of c. Then we assume that each reflection coefficient $\gamma_k = c_k e^{iu_k}$ is a random complex number with c_k and u_k independent, c_k with some distribution on $.048 \pm .018$ and μ_k uniform on $(0,2\pi)$. We then need $E(c_k^2/(1 - c_k^2)]$. According to section 5, we have upper and lower bounds:

$$\frac{.048^2}{1 - .048^2} \leq E[c_k^2/(1 - c_k^2)] \leq \frac{1}{2}[\frac{.030^2}{1 - .030^2} + \frac{.066^2}{1 - .066^2}]$$

or

$$.0023 \leq E[c_k^2/(1 - c_k^2)] \leq .0026.$$

Thus if $\eta = \eta_1 + \eta_2 + \ldots + \eta_n$, $\eta_k = \gamma_k/(1 - \gamma_k)$, then $\eta = x + iy$ has components that are (approximately) independent normal random variables with zero means and common variance

$$\sigma^2 = \tfrac{1}{2}\Sigma E[c_k^2/(1 - c_k^2)] \le \tfrac{1}{2}(8)(.0026) = .0104.$$

We take $\sigma = .1$, and the magnitude of γ, the resulting reflection coefficient, has distribution

$$P[|\gamma| < t] = P[(x - t^2/(1 - t^2))^2 + y^2 \le (t/(1 - t^2))^2]$$

or

$$P[|\gamma| < t] = C(10t^2/(1 - t^2), 10t/(1 - t^2)).$$

Figure 2 gives the distribution and density function of $|\gamma|$ for this example. It is clear that the resulting value of $|\gamma|$ is virtually certain to be much better than the pessimistic value of .36 obtained by multiplying the VSWR's - the probability that $|\gamma| < .21$ is about .90, and half the time $|\gamma|$ will be less than .12.

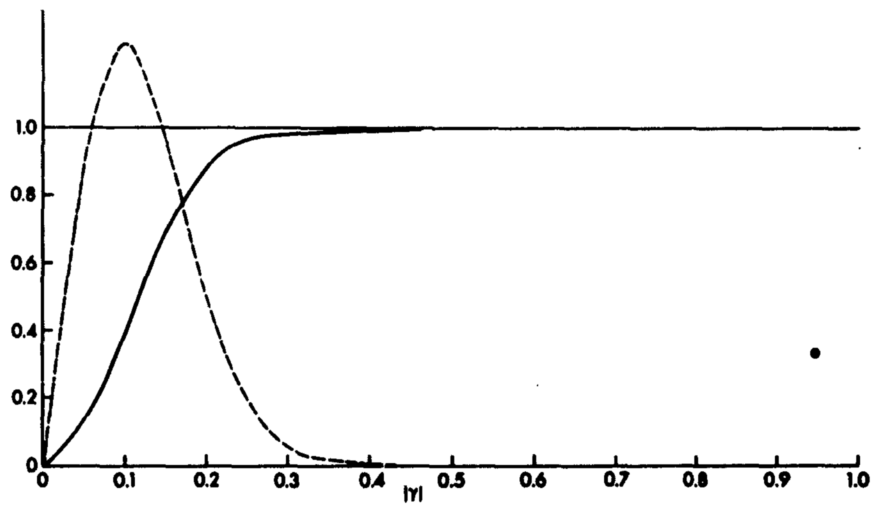

Figure 2. Distribution and density function of $|\gamma|$, Example 1.

Example 2. Suppose we have 15 discontinuities, 11 with $c = .05 \pm .02$ and 4 with $c = .10 \pm .04$. Also, suppose an additional, final, discontinuity at the antenna has complex reflection coefficient known to be $.5e^{.25\pi i}$. What can we say about total reflection losses in the system? For c's in the range $.05 \pm .02$, we have

$$\frac{.05^2}{1 - .05^2} \leq E\left(\frac{c^2}{1 - c^2}\right) \leq \frac{1}{2}\left[\frac{.03^2}{1 - .03^2} + \frac{.07^2}{1 - .07^2}\right]$$

or

$$.0025 \leq E\left(\frac{c^2}{1 - c^2}\right) \leq .0029$$

and for c's in the range $.10 \pm .04$, we found, in section 5,

$$.0101 \leq E\left(\frac{c^2}{1 - c^2}\right) \leq .0118.$$

Thus the generalized admittance, $\eta = \eta_1 + \eta_2 + \ldots + \eta_{15}$, the sum of the random admittances of the 15 discontinuities, has the form $\eta = x + iy$, where x and y are independent, normal, zero means, common variance

$$\sigma^2 \leq (11)(.5)(.0029) + 4(.5)(.0118) = .0396.$$

We thus take $\sigma = \sqrt{.0396} \cong .2$.

To $\eta = x + iy$ we must add the generalized admittance of the last discontinuity, which is $.5e^{.25\pi i}/(1 - .5e^{.25\pi i})$, or, in rectangular coordinates, $-.191 + .651i$.

Thus the generalized admittance of the entire 16 discontinuities is

$$\tau = x + .191 + (y + .651)i$$

and if $\gamma = \tau/(1 + \tau)$ is the overall reflection coefficient,

$$P[|\gamma| < t] = P[(x + .191 - t^2/(1 - t^2))^2 + (y + .651)^2 \leq (t/(1 - t^2))^2]$$

or

$$P[|\gamma| < t] = P[(\frac{x}{\sigma} + \frac{.191}{\sigma} - \frac{t^2}{\sigma(1 - t^2)})^2 + (\frac{y}{\sigma} + \frac{.651}{\sigma})^2 \leq (\frac{t}{\sigma(1 - t^2)})^2].$$

The last expression is the standard normal probability measure of a circle of radius $t/\sigma(1 - t^2)$, center distance $\sigma^{-1}[(.191 - \frac{t^2}{1 - t^2})^2 + (.651)^2]^{1/2}$ from the origin, i.e., with $\sigma = .2$,

$$P[|\gamma| < t] = C(d,r)$$

with

$$r = 5t/(1 - t^2), \; d = 5\{[.191 - t^2/(1 - t^2)]^2 + (.651)^2\}^{1/2}.$$

Figure 3 gives the distribution and density functions of $|\gamma|$ for this example.

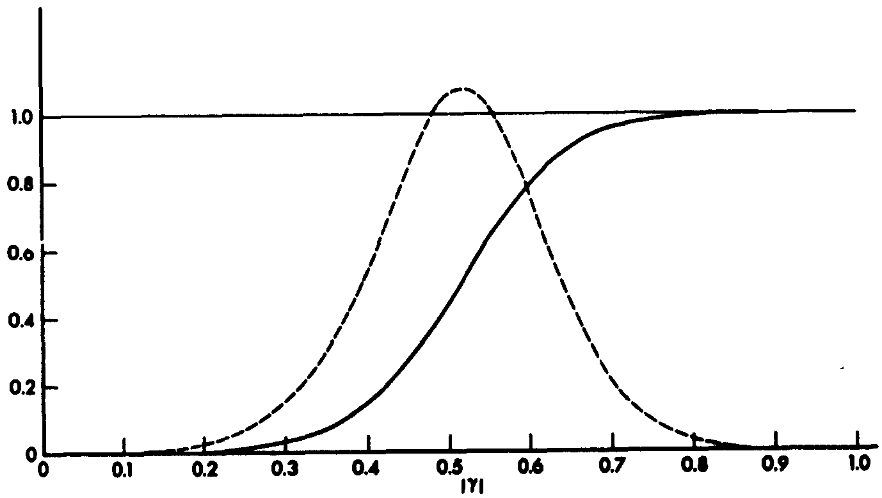

Figure 3. Distribution and density of $|\gamma|$, Example 2.

APPENDIX

Theorem 1.

Let $\gamma = ce^{iu}$ *be a complex random variable in polar form, where* c *and* u *are independent random variables, and* c *has some distribution in the interval* $(0,1)$.

If u *is symmetrically distributed over the interval* $(0,2\pi)$, *then the real and imaginary parts of* $\eta = \gamma/(1-\gamma)$ *are uncorrelated.*

If we compute the real and imaginary parts of η directly, we have

$$\eta = \gamma/(1-\gamma) = x + iy = \frac{c \cos u - u^2}{1-2c \cos u+c^2} + \frac{ic \sin u}{1-2c \cos u+c^2},$$

and it is not evident that x and y are uncorrelated. However, if we write $\eta = \gamma/(1-\gamma) = \gamma + \gamma^2 + \gamma^3 + \ldots$ then

$$(1)\quad \eta = (c \cos u + c^2 \cos 2u + c^3 \cos 3u + \ldots) + i(c \sin u + c^2 \sin 2u + c^3 \sin 3u + \ldots),$$

and we may take expectations term by term to get

$$(2)\qquad \begin{aligned} E(x) &= E(c)E(\cos u) + E(c^2)E(\cos 2u) + E(c^3)E(\cos 3u) + \ldots \\ E(y) &= E(c)E(\sin u) + E(c^2)E(\sin 2u) + E(c^3)E(\sin 3u) + \ldots \\ E(xy) &= \sum_{rs}\sum E(c^{r+s})E(\cos ru \sin su). \end{aligned}$$

Now if u is symmetrically distributed over $(0,2\pi)$ then $E(\sin ru) = E(\sin ru \cos su) = 0$. Hence $E(y) = 0$ and $E(xy) = 0$, so that x and y are uncorrelated.

Of particular interest is the case where u is uniformly distributed:

Theorem 2.

If $\gamma = ce^{iu}$, where c and u are independent, c has some distributions on the interval $(0,1)$, and u is uniformly distributed on the interval $(0,2\pi)$, then the real and imaginary parts of $\eta = \gamma/(1 - \gamma) = x + iy$ are uncorrelated, with expected values

$$E(x) = E(y) = E(xy) = 0$$

and variances

$$E(x^2) = E(y^2) = \frac{1}{2}E\left[\frac{c^2}{1 - c^2}\right] .$$

The proof of Theorem 1 applies, except that we must compute the variances of x and y. Squaring and taking expected values of the series expressions for x or y, the cross product terms are zero, and there remains, since $E(\sin^2 ru) = E(\cos^2 ru) = \frac{1}{2}$,

$$E(x^2) = E(y^2) = \frac{1}{2}[E(c^2) + E(c^4) + E(c^6) + \dots]$$

$$= \frac{1}{2}E\left[\frac{c^2}{1 - c^2}\right].$$

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REFERENCES

[1] G. Marsaglia, Tables of the Normal Probability Measure of an Offset Circle , Boeing Scientific Research Laboratories.

[2] C. G. Montgomery, R. H. Dicke, E. M. Purcell, Principles of Microwave Circuits, Radiation Laboratory Series No. 8, McGraw Hill, New York, 1948.

[3] J. A. Mullen and W. L. Pritchard, The Statistical Prediction of Voltage Standing Wave Ratio. IRE Transactions on Microwave Theory and Techniques, Vol. 5, p. 127 - 130.

[4] G. L. Ragan, Microwave Transmission Circuits, Radiation Laboratory Series No. 9, McGraw Hill, New York, 1948.

[5] The Rand Corporation, Offset Circle Probabilities, Report R-234 (ASTIA NO. AD 93608), 1952.

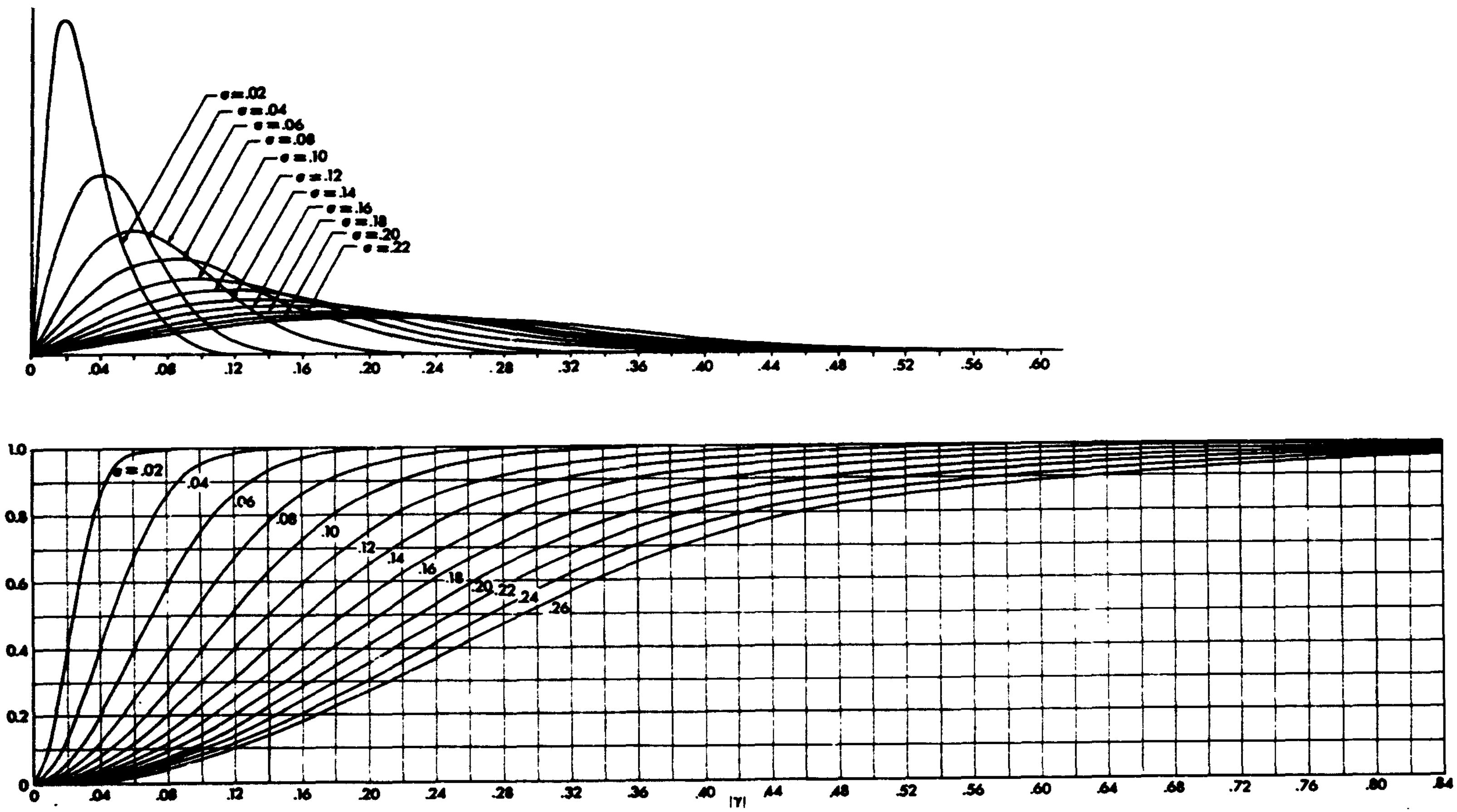

Figure 1. Distribution and density functions of $|\gamma|$, the magnitude of the overall reflection coefficient, for various values of σ.